Pathologies of the volume function

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Iitaka dimension

- X smooth projective variety, D a divisor.
- Theorem: There are constants C_i such that for large divisible m, $C_1 m^\kappa < h^0(X, mD) < C_2 m^\kappa$.
- This κ is the Iitaka dimension of D.

Numerical invariance

- This is not a numerical invariant: it can happen that $D \equiv D'$ but different Iitaka dimension.
- On threefold: two numerically equivalent divisors, one rigid and one which moves in a pencil.
- We want a numerically invariant version ν

Numerical dimension: sections ([Nakayama])

- Fix sufficiently ample A.
- Look at growth of $h^0(\lfloor mD \rfloor + A)$ as m increases.
- How does it behave?

Nakayama lemma

- Thm (Nakayama): If $h^0(\lfloor mD \rfloor + A)$ is not bounded in m, then $h^0(\lfloor mD \rfloor + A) > Cm$ for some C.

Proposition 3.3.2. *Let X be a smooth projective variety and let D be a pseudo-effective $\mathbb{R}$-divisor. Let B be any big $\mathbb{R}$-divisor.*

If D is not numerically equivalent to $N_\sigma(D)$, then there is a positive integer k and a positive rational number β such that

$$h^0(X, \mathcal{O}_X(\llcorner mD \lrcorner + \llcorner kB \lrcorner)) > \beta m, \qquad \text{for all} \qquad m \gg 0.$$

Proof. Let A be any integral divisor. Then we may find a positive integer k such that

$$h^0(X, \mathcal{O}_X(\llcorner kB \lrcorner - A)) \geq 0.$$

Thus it suffices to exhibit an ample divisor A and a positive rational number β such that

$$h^0(X, \mathcal{O}_X(\llcorner mD \lrcorner + A)) > \beta m \qquad \text{for all} \qquad m \gg 0.$$

Replacing D by $D - N_\sigma(D)$, we may assume that $N_\sigma(D) = 0$. Now apply (V.1.12) of [28]. □

Background

Question (Nakayama, 2002)

Suppose that D is a pseudoeffective divisor and that A is ample. Then there exist constants C_1, C_2 and a positive integer $\nu(D)$ so that:

$$C_1 m^{\nu(D)} \leq h^0(\lfloor mD \rfloor + A) \leq C_2 m^{\nu(D)}$$

Volume

- Volume of D is $\lim_{m\to\infty} \frac{h^0(mD)}{m^d/d!}$.
- $\mathrm{vol}(D + tA)$ for small $t \leftrightarrow h^0(mD + A)$ for large m:

$$\mathrm{vol}\left(D + \frac{1}{m}A\right) = \frac{1}{m^3}\mathrm{vol}(mD + A) \approx \frac{1}{m^3}h^0(mD + A)$$

Main result

- For each $N \geq 3$ there exists a smooth Calabi–Yau N-fold such that for any $\delta \in \left[1, \frac{N}{2}\right]$ one can find a pseudoeffective $\mathbb{R}$-divisor D with:

$$\limsup_{m\to\infty} \frac{\log h^0(X, \lfloor mD \rfloor + A)}{\log m} = N - \delta$$

$$\liminf_{m\to\infty} \frac{\log h^0(X, \lfloor mD \rfloor + A)}{\log m} = \frac{N}{2}$$

$$\liminf_{s\to 0^+} \frac{\log \operatorname{vol}(D + sA)}{\log s} = \delta,$$

$$\limsup_{s\to 0^+} \frac{\log \operatorname{vol}(D + sA)}{\log s} = \frac{N}{2}$$

Regularity of volume

- Volume is C^1 on the big cone, but not C^2 in general.
- But what about the pseudoeffective boundary? Could $s \mapsto \mathrm{vol}(D + sA)$ have extra regularity?
- The example: $s \mapsto \mathrm{vol}(D + sA)$ is C^1 but not $C^{1,\alpha}$ on $[0, \epsilon)$ for any $\alpha > 0$.
- (Could it be $C^{1,1}_{loc}$ inside the big cone?)

Warm-up

- Let X be $(1,1)$, $(1,1)$, $(2,2)$ complete intersection in $\mathbb{P}^3 \times \mathbb{P}^3$.
- This is a smooth CY3, Picard rank 2.
- Studied by Oguiso in connection with Kawamata–Morrison conj.

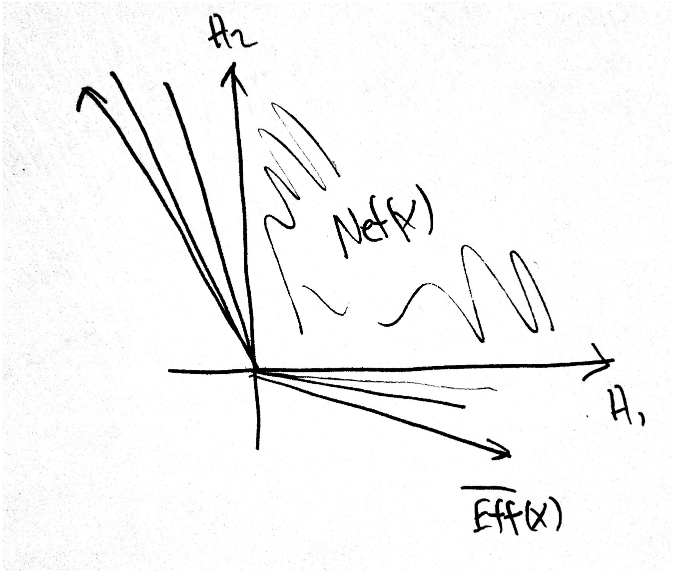
The example

- It has some birational automorphisms coming from covering involutions.
- Action on $N^1(X)$ given by

$$\tau_1^* = \begin{pmatrix} 1 & 6 \\ 0 & -1 \end{pmatrix}, \quad \tau_2^* = \begin{pmatrix} -1 & 0 \\ 6 & 1 \end{pmatrix}, \quad \phi^* = \begin{pmatrix} 35 & 6 \\ -6 & -1 \end{pmatrix}$$

- Composition has infinite order: $\lambda = 17 + 12\sqrt{2}$.
- Nef cone bounded by H_1, H_2.
- Psef cone bounded by $(1 \pm \sqrt{2})H_1 + (1 \mp \sqrt{2})H_2$.
- Let $D_+ = c_1 H_1 + c_2 H_2$ be divisor in this class.

Cones



The trick

- For any line bundle whatsoever on X, you can compute $h^0(D)$.
- Pull it back some number of times, it's ample, and then compute h^0 for ample using Riemann-Roch+Kodaira vanishing!
- HRR on CY3:

$$\chi(D) = \frac{D^3}{6} + \frac{D \cdot c_2(X)}{12}.$$

Let's compute

- Suppose our ample is $A = M_1 D_+ + M_2 D_-$.
- We need to compute $h^0(\lfloor mD \rfloor + A)$.
- How many times to pull back? Looks like a mess, but there's an invariant quadratic form: the product of the coefficients when you work in the eigenbasis.

$$\phi^* = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$$

Let's compute, II

- $mD_+ + A = (m + M_1)D_+ + M_2 D_-$.
- The pullback that's ample has the two coefficients roughly equal, about

$$\left(\sqrt{(m+M_1)M_2}\right) D_+ + \left(\sqrt{(m+M_1)M_2}\right) D_-$$

- Then $h^0(\lfloor mD_+ \rfloor + A) \approx Cm^{3/2}$.

Extensions of the computation

- We used the fact that D_+ and D_- that span an eigenspace intersecting the ample cone.
- In this case, an eigenvector always has $\nu_{vol}(D_+) = \frac{\dim X}{2}$ if $\lambda_1(f) = \lambda_1(f^{-1})$.

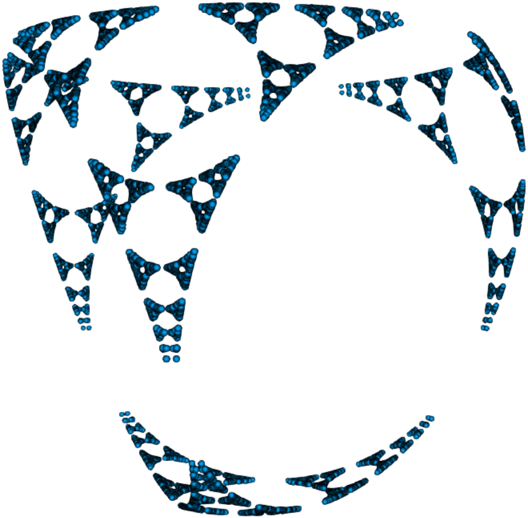
Another example

- Let X be a $(2,2,2,2)$ hypersurface in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$; key aspects of geometry worked out by Cantat–Oguiso.
- It's a smooth CY3.
- $(\mathbb{Z}/2\mathbb{Z})^{*4} \subset \mathrm{Bir}(X)$ coming from covering involutions.
- Kawamata–Morrison conjecture is true, so we can still compute volume of any class very easily in principle...

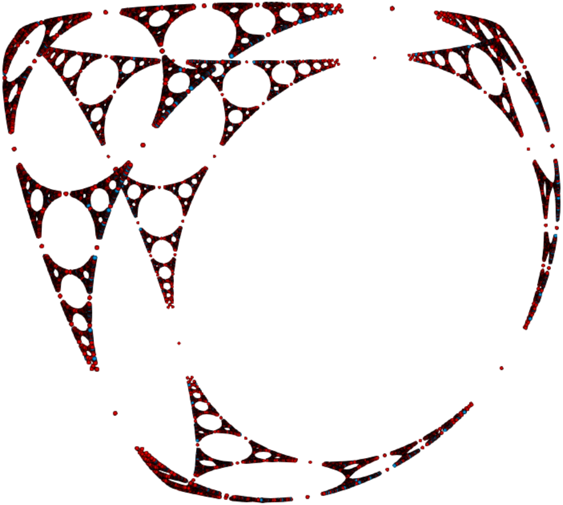
Two kinds of divisors on the psef boundary

- There are some distinguished classes:
  - eigenvectors, which all have $\mathrm{vol}(D + tA) \sim Ct^{3/2}$;
  - semiample $\pi_i^*(\mathcal{O}_{\mathbb{P}^1}(1)) + \pi_j^*(\mathcal{O}_{\mathbb{P}^1}(1))$, plus their orbits under $\mathrm{Bir}(X)$, which all have $\mathrm{vol}(D + tA) \sim Ct$.

First picture: the eigenvectors



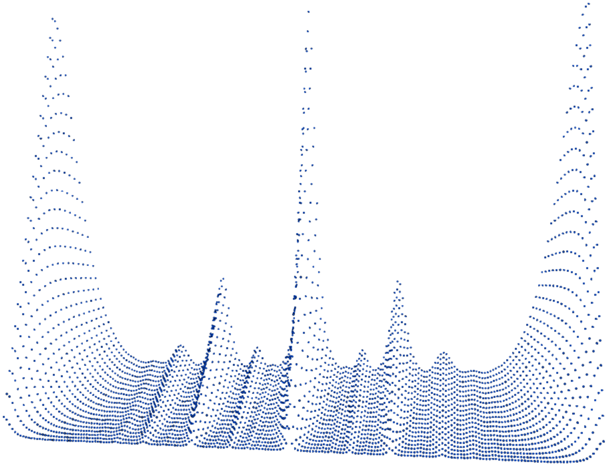
Second picture: the semiample type



Volume near the boundary

- There are some "circles" on the boundary of $\overline{\mathrm{Eff}}(X)$ on which both eigenvectors and semiample type are dense.
- The former have $\mathrm{vol}(D + tA) \sim Ct^{3/2}$, the latter have $\mathrm{vol}(D + tA) \sim Ct$.
- How is this possible?
- Because the volume function is so easy to compute numerically, we can plot it!

Volume near the boundary



A quotient of the movable cone

- Action of $\mathrm{Bir}(X)$ preserves a quadratic form of signature $(1,3)$ on $N^1(X)$ (Cantat–Oguiso).
- Restricting to classes of norm 1 and taking quotient of movable cone by this action, we obtain action of $\mathrm{Bir}(X)$ on a (non-compact) hyperbolic 3-manifold Σ.
- Volume function descends to $\overline{\mathrm{vol}} : \Sigma \to \mathbb{R}_{\geq 0}$.

How to imagine these classes

- Paths $D + sA$ in $N^1(X)$ determine a geodesic on Σ after normalization:

$$\gamma(s) = \frac{D + sA}{\sqrt{Q(D + sA, D + sA)}} \approx \frac{D + sA}{\sqrt{s}}$$

- Then

$$\mathrm{vol}(D + sA) = \mathrm{vol}\left(\sqrt{s}\gamma(s)\right) = s^{3/2}\,\mathrm{vol}(\gamma(s)) = s^{3/2}\overline{\mathrm{vol}}([\gamma(s)])$$

- When $[\gamma(s)]$ is near the middle of Σ, we see $s^{3/2}$ behavior, but volume gets larger when the geodesic goes out a cusp.

Recurrent geodesics

- If geodesic stays in a compact region (typical, e.g. eigenvectors), we see $s^{3/2}$ growth as $s \to 0$.
- But if a geodesic wanders out a cusp, we see the larger volume s^1.
- Every geodesic ray is either returns infinitely often to a compact set, or goes into the cusp.
- In particular, we either see $\mathrm{vol}(D + sA) \sim s$ growth, or $\mathrm{vol}(D + sA) \sim Cs^{3/2}$ along an infinite subsequence of s.

Cusp excursions

- Suppose ℓ_i is a sequence of (sufficiently large) positive reals. For any $x_0 \in \mathcal{M}^{cc}$ and open $U \subset T_{x_0}M$ three is an infinite geodesic ray γ starting
  - initial tangent vector is in U;
  - $\gamma = \bigcup [x_i, x_{i+1})$ with $\ell_{[x_i, x_{i+1})} = \ell_i + O(1)$ and $d(x_0, x_i) = O(1)$;
  - $d(x_0, -)$ on (x_i, x_{i+1}) is roughly linearly growth out to $\frac{1}{2}\ell_i$ and then decreasing back.
- The key technical ingredient is "gluing geodesics": we write down each desired cusp excursion separately, and as long as the endpoint data are very close, there is a nearby geodesic approximating the union.

From volume to sections

- We get oscillation of $\mathrm{vol}(D + sA)$, hence $m^3 \,\mathrm{vol}\left(D + \frac{1}{m}A\right)$ if we make sure oscillations occur when $s = \frac{1}{m}$.
- We also need to bound errors of (a) $h^0(X, \lfloor mD \rfloor + A)$ vs $\mathrm{vol}(\lfloor mD \rfloor + A)$ and (b) $\mathrm{vol}(\lfloor mD \rfloor + A)$ vs $\mathrm{vol}(mD + A)$.
- The first is fairly easy since we can compute h^0 of the ample pullback using HRR; one term is the volume and we bound the error.
- For the second we need to check how far rounding moves in the hyperbolic distance and make sure it doesn't interfere.
- (Both are OK.)

Nakayama's κ_σs

$$\kappa_\sigma^+(D) = \min\left\{k : \limsup_{m\to\infty} \frac{h^0(\lfloor mD \rfloor + A)}{m^k} < \infty\right\}$$

$$\kappa_\sigma(D) = \max\left\{k : \limsup_{m\to\infty} \frac{h^0(\lfloor mD \rfloor + A)}{m^k} > 0\right\}$$

$$\kappa_\sigma^-(D) = \max\left\{k : \liminf_{m\to\infty} \frac{h^0(\lfloor mD \rfloor + A)}{m^k} > 0\right\}$$

In our example:

$$\kappa_\sigma^{\mathbb{R},-}(D) = \left\lfloor \frac{N}{2} \right\rfloor, \quad \kappa_\sigma^{\mathbb{R}}(D) = \kappa_\sigma^{\mathbb{R},+}(D) = N-1$$

So these things are not the same.

A question of Fujino

- Let X be smooth projective over $\mathbb{C}$ and D pseudoeffective. Then there exist $m_0 \geq 1$, $c > 0$, and ample A so that

$$h^0(X, \lfloor mm_0 D \rfloor + A) \geq cm^{\kappa_\sigma(D)}$$

- We show that this fails starting in dimension 5.